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Spherical q-functions

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Abstract. Explicit realization of $SU_q(2)$ algebra is found in terms of operators acting on a two-dimensional sphere and depending on (θ, φ) variables. The corresponding eigenfunctions ('spherical q-functions') appear to be a q-generalization of ordinary Legendre polynomials. The recurrence relations and explicit expression in terms of Askey–Wilson polynomials are obtained. The weight function of these polynomials is expressed via double-periodic elliptic functions.

1. Introduction

Recent global investigations of the quantum groups and algebras, especially the $SU_q(2)$ algebra, are mostly concentrated around general problems such as spectra, multiplet structure, etc. The problem of explicit expression for representation functions still escapes the attention of specialists in the field. The situation is still more astonishing because historically the representations of the rotation group were discovered before the group structure itself was established.

Here, we would like to fill this gap, dealing with spherical functions for the $SU_q(2)$ algebra rather than the $SU_q(2)$ group.

The commutation relations for this algebration are [1]

$$[J_0, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = (\sinh 2\omega J_0)/\sinh \omega. \tag{1.1}$$

In what follows we shall assume that $\omega > 0$.

The representations are defined by the equations

$$J_0 \psi_{jm} = m\psi_{jm} \qquad J_q^2 \psi_{jm} = -\lambda_j \psi_{jm} \tag{1.2a}$$

where

$$J_q^2 = J_- J_+ + (\cosh \omega(2J_0 + 1))/(2\sinh^2 \omega) \tag{1.2b}$$

is the Casimir operator of $SU_q(2)$ and λ_j is its eigenvalue.

These are all the preliminaries needed.

2. Choice of realization

We are going to construct the reps on the two-dimensional sphere with ordinary metrics in Hilbert space given as

$$\langle\Phi|\psi\rangle=\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\sin\theta\,\mathrm{d}\theta\,\Phi^*(\theta,\varphi)\psi(\theta,\varphi). \tag{2.1}$$

In what follows it will be assumed that variables are separated

$$\psi_{jm}(\theta,\varphi)=\mathrm{e}^{\mathrm{i}m\varphi}Y_{jm}(\theta). \tag{2.2}$$

Consequently the operators J_k have the form

$$J_0=\mathrm{i}\partial_\varphi \qquad J_+=A(a^+U-aV^{-1}) \qquad J_-=A(aV-a^+U^{-1}) \tag{2.3}$$

where the operators a and a^+ are mutually commuting with U and V and A is a normalization constant.

The commutators (1.1) are identically satisfied if

$$\begin{gathered}[J_0,U]=U \qquad [J_0,V]=-V \qquad UV=VU\exp(2\omega)\\ 2A^2\sinh\omega\;UV=\exp(\omega(1+2\mathrm{i}\partial_\varphi) \qquad [a^+,a]_\omega=-\exp(\omega).\end{gathered} \tag{2.4}$$

Here we have used the so-called 'q-mutator'

$$[L,M]_\omega=\exp(\omega)LM-\exp(-\omega)ML. \tag{2.5}$$

Equations (2.4) are easily solved giving the result

$$U=\exp(\mathrm{i}\varphi+\mathrm{i}\omega\partial_\varphi) \qquad V=\exp(-\mathrm{i}\varphi+\mathrm{i}\omega\partial_\varphi). \tag{2.6}$$

We then obtain the appropriate normalization factor

$$A=(\exp(2\omega)-1)^{-1/2}.$$

Noting that the quantum number m is an integer of any sign and that $|m|\leqslant j$, then, from (1.2*b*), we have

$$\lambda_j=(\cosh\omega(2j+1))/(2\sinh^2\omega). \tag{2.7}$$

The next step is a construction of operators a^+ and a in terms of lattitude variable θ. Working in the same manner

$$a=B(\theta)(T+T^{-1}) \qquad a^*=B(\theta)(W+W^{-1}) \tag{2.8}$$

and we obtain

$$\begin{gathered}T=\exp(\mathrm{i}\theta+\mathrm{i}\omega\partial_\theta) \qquad W=\exp(-\mathrm{i}\theta+\mathrm{i}\omega\partial_\theta)\\ B(\theta)=1/[2(1-\mathrm{e}^{-2\omega})^{1/2}\sin\theta].\end{gathered} \tag{2.9}$$

It may be verified that for $\omega\to 0$ we return to the standard realization of angular momentum via the differential operators [2]

$$J_\pm=\exp(\pm\mathrm{i}\varphi)(\pm\partial_\theta+\mathrm{i}\cot\theta\,\partial_\varphi). \tag{2.10}$$

In our case, due to exponentiation of ∂_φ, ∂_θ we are really dealing with *difference* operators. For example, the U-operator has the following displacement property

$$Uf(\varphi)=\exp(\mathrm{i}\varphi-\omega/2)f(\varphi+\mathrm{i}\omega) \tag{2.11}$$

along with corresponding ones for V, T, W. Thus, parameter ω is playing the role of 'quantization' agent in the imaginary direction of the complex φ-plane. Instead of the line, we are running rather on the lattice in this plane. The origin of these properties may be traced further, right up to commutation relations (1.1). These imply not only anisotropy in representation space but also a cellular structure.

Note that the last commutation relations (2.4) imply that the operators a and a^+ are the annihilaton and creation operators for a q-oscillator [3]. Thus the latitude dependence of the operators J_k (i.e. θ-dependence) is provided by a one-mode q-oscillator. These results may be compared with the approach of Macfarlane [3] where onlyφ-dependence of the operators J_k was found. It is surprising that there are no classical analogues of this q-oscillator representation because the operators a and a^+ as given by the formulae (2.8)–(2.9) disappear in the classical limit, whereas the operators J_k have the correct classical limit (2.10).

3. 'Vertical' recurrences

Equation (1.2)

$$J_-J_+\psi_{jm}=(\lambda_j-\lambda_m)\psi_{jm}=[j-m][j+m+1]\psi_{jm} \tag{3.1}$$

$$[x]=\sinh\omega x/\sinh\omega \tag{3.1a}$$

may be written in the form

$$J_+\psi_{jm}=\alpha_{jm}\psi_{j,m+1} \qquad J_-\psi_{j,m+1}=\beta_{jm}\psi_{jm}. \tag{3.2}$$

The raising J_+ and lowering J_- operators are Hermitian conjugated in the metric (2.1) so their matrix elements are conjugated also: $\alpha_{jm}=\beta^*_{jm}$. Discarding the phase, we have [1]

$$\alpha_{jm}=\beta_{jm}=([j-m][j+m+1])^{1/2}. \tag{3.3}$$

$\psi_{jm}(\theta,\varphi)$, being eigenfunctions of the Hermitian operator, are orthogonal to each other in the sense of (2.1). Extracting from them some factors (see below) one can reduce $\psi_{jm}(\theta,\varphi)$ to polynomials being a certain generalization of Legendre polynomials. There are many possible ways to generalize them [4] but we arrive at one suitable for use in equations of the algebra $SU_q(2)$.

Equation (3.2) allows one to derive all the needed functions starting from the initial one. Using the displacement properties of the operators U, V, T, W we obtain

$$([j-m][j+m+1])^{1/2}\,Y_{j,m+1}(\theta)=(2\mathrm{i}\sinh\omega\sin\theta)^{-1}$$

$$Y_{jm}(\theta+\mathrm{i}\omega)\sin(\theta-\mathrm{i}\omega m)-Y_{jm}(\theta-\mathrm{i}\omega)\sin(\theta+\mathrm{i}\omega m). \tag{3.4}$$

This and related relations preserve the quantum number j and may be termed 'vertical' recurrences in view of their geometrical meaning (see figure 1).

This mathematical structure of (3.4) is rather complicated—it involves a simulta-

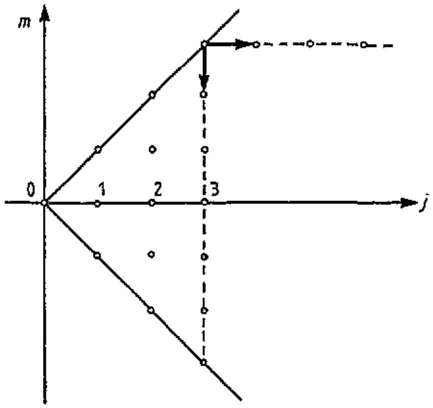

Figure 1.

neous alteration of argument θ and index m. It may be simplified by extracting from Y_{jm} normalization factors

$$Y_{jm}(\theta)=C_{jm}Y_{mm}(\theta)Q_{jm}(\theta). \tag{3.5}$$

In what follows we shall assume that $m\geqslant 0$, so $j\geqslant m$. Choosing

$$C_{jm}=C_{j0}([j-m]!/(j+m]!)^{1/2} \tag{3.6}$$

we can get rid of radicals so that

$$Q_{j,m+1}(\theta)=\frac{Q_{jm}(\theta+\mathrm{i}\omega)\sin(\theta-\mathrm{i}\omega m)Y_{mm}(\theta+\mathrm{i}\omega)-(\omega\to-\omega)}{2\sinh\omega\sin\theta\, Y_{m+1,m+1}(\theta)}. \tag{3.7}$$

The function $Y_{mm}(\theta)$ is the 'highest' in the corresponding multiplet and may be found from the equaiton $J_+\psi_{mm}=0$ or, according to (3.4)

$$Y_{mm}(\theta+\mathrm{i}\omega)\sin(\theta-\mathrm{i}\omega m)=Y_{mm}(\theta-\mathrm{i}\omega)\sin(\theta+\mathrm{i}\omega m). \tag{3.8}$$

The solution of this functional equation is written at once as

$$Y_{mm}(\theta)=G_m(\theta)\prod_{k=0}^{m-1}\sin(\theta+\mathrm{i}\omega(m-1-2k)) \tag{3.9}$$

where $G_m(\theta+\mathrm{i}\omega)=G_m(\theta-\mathrm{i}\omega)$ is an arbitrary function with period $2\mathrm{i}\omega$.

Suppose that $G_m(\theta)=g(\theta+\mathrm{i}\omega m)$, then we reduce (3.7) to the simplest form

$$Q_{j,m+1}(\theta)=\frac{Q_{jm}(\theta+\mathrm{i}\omega)-Q_{jm}(\theta-\mathrm{i}\omega)}{2\mathrm{i}\sinh\omega\sin\theta}. \tag{3.10}$$

This is the difference equation, replacing the usual connection

$$P_{j,m+1}(\cos\theta)=-P'_{jm}(\cos\theta)$$

for ordinary associated Legendre polynomials.

Defining the difference operator Δ

$$\Delta f(x)=f(x+\mathrm{i}\omega)-f(x-\mathrm{i}\omega) \tag{3.11}$$

equation (3.10) becomes

$$Q_{j,m+1}(\theta)=-\Delta Q_{jm}(\theta)/\Delta\cos\theta \tag{3.12}$$

in complete similarity to the above-mentioned connection for P_{jm}.

Thus, proceeding from $Q_{jm}(\theta)$, the whole multiplet may be recovered, for our purposes, but we need the relation connecting the functions Q_{j0} themselves. This relation should be of a 'horizontal' type and must be treated in a completely different manner.

4. 'Horizontal' recurrences

The origin of 'vertical' recurrences (3.4) is the existence of the operators $J_\pm$ conserving j and changing m. There are no suitable operators (belonging to $SU_q(2)$) that conserve m and change only j, giving rise to 'horizontal' recurrence relations.

Instead, we will exploit the relation

$$\cos\theta\,\psi_{jm} = a_{j+1,m}\,\psi_{j+1,m} + a_{jm}\,\psi_{j-1,m} \tag{4.1}$$

obtained in the appendix by means of our technique employing the AW(3) algebra [5]. The matrix elements are

$$a_{jm} = \cosh\omega j\left[\frac{[j+m][j-m]}{[2j+1][2j-1]}\right]^{1/2}. \tag{4.2}$$

As for $Q_{jm}(\theta)$, they obey the relation (taking $C_{j0}=[2j+1]^{1/2}$)

$$\begin{aligned}\cos\,\theta\,Q_{jm}(\theta) = {}& [j+1-m]/[2j+1]\,Q_{j+1,m}(\theta)\cosh\omega(j+1)\\ & + [j+m]/[2j+1]\,Q_{j-1,m}(\theta)\cosh\omega j\end{aligned} \tag{4.3}$$

of the required 'horizontal' type. For $\omega\to 0$ the usual recurrence for associated Legendre polynomials is restored [6].

Solution of (4.3) must be sought in terms of the basic hypergeometric function

$${}_4\Phi_3(a_1,\ldots,a_4;b_1,\ldots,b_3|x) = \sum_{n=0}^{\infty}\frac{\prod_{s=1}^{4}(a_s;q)_n}{\prod_{s=1}^{3}(b_s;q)_n}x^n \tag{4.4}$$

where

$$(a;q)_n = (1-a)(1-aq)\ldots(1-aq^{n-1}) \tag{4.4a}$$

Indeed, using the results of Askey and Wilson [4] we can conclude that (4.3) is satisfied by function (4.4), with the parameters being

$$\begin{aligned}&a_1 = = q^{m-j} \qquad a_2 = q^{j+m+1} \qquad a_3 = e^{i\theta}\,q^{(m+1)/2} \qquad a_4 = e^{-i\theta}\,q^{(m+1)/2}\\ &b_1 = -b_2 = -b_3 = q^{m+1} \qquad q = \exp(-2\omega) \qquad x = q.\end{aligned} \tag{4.5}$$

This result provides us with polynomial of order $N=j-m$ because the Pochhammer q-symbol $(a_1;q)_n$ vanishes for $n\geqslant N$. So

$$Q_{jm}(\theta) = {}_4\Phi_3(q^{m-j},a_2,a_3,a_4;b_1,b_2,b_3|q). \tag{4.6}$$

Normalization of the derived polynomials is related to the problem of determining the corresponding weight function $W_m(\theta)$

$$\int_0^\pi \sin\theta\,\mathrm{d}\theta\,W_m(\theta)Q_{jm}(\theta)Q^*_{j'm}(\theta) = A_{jm}\delta_{jj'} \tag{4.7}$$

with some normalization constant A_{jm}.

It is evident from (2.1) and (3.5) that the weight function is expressible through the 'highest' function

$$W_m(\theta)=|Y_{mm}(\theta)|^2. \tag{4.8}$$

We may confine ourselves to consideration of the case $m=0$ because m-dependence of $Y_{mm}(\theta)$ is completely described by the formula (3.9), and the m-dependence of the associated polynomials q_{jm} is determined by (3.10).

As was mentioned, $W_0(\theta+2i\omega)=W_0(\theta)$ and obviously $W_0(\theta+\pi)=W_0(\theta)$, i.e. $W_0(\theta)$ is a double-periodic funcition. If, additionally, one assumes that this function vanishes at the ends of the $(0, \pi)$ interval, then it is natural to take an elliptic function for it

$$W_0(\theta)=C\,\mathrm{sn}\,(2\mathrm{K}\theta/\pi)$$

where $K=K(q)$ is an elliptic integral of first kind, serving as a real period for $\mathrm{sn}(z)$. The constant C is defined by the normalization condition

$$\int_0^\pi W_0(\theta)\sin\theta\,\mathrm{d}\theta=1 \tag{4.9}$$

and is equal to $(2\kappa K/\pi^2)\sinh\omega$ where κ is the elliptic modulus. Our assumption for the weight function can be justified by results of Askey and Wilson [4] who obtained an explicit expression of the weight function for generic ${}_4\Phi_3$ polynomials. So we have finally

$$W_0(\theta)=(2\kappa K/\pi^2)\sinh\omega\,\mathrm{sn}(2K\theta/\pi). \tag{4.10}$$

In the general case $m\geqslant 0$ one obtains

$$W_m(\theta)=(2\kappa K/\pi^2)\sinh\omega\,\mathrm{sn}(2\mathrm{K}\theta/\pi+\mathrm{i}mK')$$
$$\times\prod_{k=0}^{m-1}(1-2q^{m-1-2k}\cos 2\theta+q^{2(m-1-2k)})/4 \tag{4.11}$$

($\mathrm{i}K'=2\mathrm{i}\omega K/\pi$ is the imaginary period of the elliptic functions).

Askey and Wilson state 'No facts have been found for the q-extension of Legendre polynomials. This may be because these polynomials have no special properties or because we don't known where they live' [4].

Now we do know 'where these polynomials live'—it is representation space for a q-algebra—but the origin of double-periodicity is still quite obscure.

5. Conclusion

Thus, a simple and direct question—how to obtain an explicit expression for representation functions?—received a final answer: these functions are reduced to polynomials compactly written through the basic hypergeometric function ${}_4\phi_3(q)$, whose parameters contain quantum numbers j, m and argument θ. Nearly the same might be said about ordinary Legendre polynomials: ${}_4\Phi_3$ is replaced by the Gaussian hypergeometric function ${}_2F_1$ and parameter q is fixed at 1. So the general result is more or less expected, since the parameter q was simply inserted 'by hand' into commutation relations of the $SU_q(2)$ algebra.

But a deeper insight would recognize the unusual features contained in the discussed results. These are:

(i) emergence (in all eigenfunctions) of the factor $g(\theta)=Y_{00}(\theta;\omega)$, being a double-periodic function of θ;

(ii) radical changes of topology of the complex θ-plane—it becomes a torus instead of uniform plane.

Both properties are tightly bound together and give rise to the problem of their origin. Perhaps the answer is that the function $\sinh 2\omega J_0$ in the commutation relations (1.1) generates finite shifts of the required size in the imaginary direction. In turn, this could lead to a corresponding imaginary period in the complex θ-plane. It is interesting to note that Macfarlane [3] found that the vacuum state of a q-oscillator reduces to some theta-function; however, the origin of it is, as yet, unclear.

Finally, we would like to stress that our treatment of the spherical q-function is quite different from previous approaches [7], where the corresponding q-functions depend on non-commuting variables beloning to the $SU_q(2)$ group. In particular, our metric (2.1) coincides with the standard 'classical' one for a two-dimensional sphere, in contrast to all previous treatments dealing with rather exotic 'quantum spheres'. Such (θ, φ) parametrization of spherical q-functions is most attractive for the purposes of physical applications.

One can expect that the results obtained will be useful in those probelms where the quantum algebras appear to be a dynamical symmetry of the considered system—lattice models, quantum Hall effects, etc.

Appendix

Here, we explicity obtain the 'horizontal' recurrence relations (4.1) and (4.2). For this, we use the technique of AW(3) algebra [5], this being a powerful tool for different q-problems.

Let us introduce two operators

$$K_1=J_q^2 \qquad K_2=\cos\theta. \tag{A.1}$$

It can be directly verified by means of the explicit realization (2.3) that K_1, K_2 together with their q-mutator (2.5) form the algebra

$$[K_1, K_2]_\omega=K_3 \qquad [K_2, K_3]_\omega=C_1K_1+D \qquad [K_3, K_1]_\omega=C_2K_2 \tag{A.2}$$

with structure constants

$$C_1=\sinh^2 2\omega \qquad C_2=\coth^2\omega \qquad D=-2\cosh\omega\,\sinh^2\omega\, m. \tag{A.3}$$

Recall that the quantum number m is assumed to be fixed ('horizontal' recurrence) as an eigenvalue of the operator J_0 commuting with both K_1 and K_2.

The algebra (A.2) is a special case of the so-called Askey–Wilson algebra AW(3) introduced in [5]. Its Casimir operator $\hat{Q}$ commuting with all K_i is

$$\hat{Q}=\{K_3, \tilde{K}_3\}/2+\cosh 2\omega(C_1K_1^2+C_2K_2^2)+2D\cosh^2\omega\, K_l \tag{A.4}$$

where $\{\ldots\}$ stands for anticommutator and

$$\tilde{K}_3=[K_1, K_2]_{-\omega}.$$

Given the realization (A.1) and (2.3), Casimir $\hat{Q}$ takes the value

$$Q=\coth^2\omega(\sinh^2\omega+\cosh 2\omega\, m). \tag{A.5}$$

A specific feature of AW(3) is that the operator K_2 is three-diagonal in discrete basis ψ_{jm} of the operator K_1. Namely

$$K_2\psi_{jm}=a_{j+1,m}\psi_{j+1,m}+a_{jm}\psi_{j-1,m} \tag{A.6}$$

where the coefficients a_{jm} are directly obtained from AW(3) as [5]

$$a_{jm}^2=\frac{C_1\lambda_j\lambda_{j-1}+D(\lambda_j+\lambda_{j-1})-Q}{(\lambda_{j+1}-\lambda_{j-1})(\lambda_j-\lambda_{j-2})} \tag{A.7}$$

with λ_j being the eigenvalues of K_1 (see (2.7)). Inserting explicit expressions for λ_j, C_i, D and Q into (A.7) one obtains

$$a_{jm}^2=\cosh^2\omega\, j[j+m][j-m]/[2j+1][2j-1] \tag{A.8}$$

which is the required formula (4.2) for the recurrence coefficients.

Note that AW(3) determines a_{jm} only up to an arbitrary phase factor which may be connected with that of ψ_{jm}.

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